

Perceiving the Infinite and the Infinitesimal World: Unveiling and Optical Diagrams in the Construction of Mathematical Concepts

Lorenzo Magnani (lmagnani@cc.gatech.edu)

University of Pavia, Pavia, Italy
and Georgia Institute of Technology, Atlanta, GA, USA

Riccardo Dossena (rdossena@pure-reason.unipv.it)

University of Pavia, Pavia, Italy

Abstract

Many important concepts of the calculus are difficult to grasp, and they may appear epistemologically unjustified. For example, how does a real function appear in “small” neighborhoods of its points? How does it appear at infinity? Diagrams allow us to overcome the difficulty in constructing representations of mathematical critical situations and objects. For example they actually reveal the behavior of a real function not “close to” a point (as in the standard limit theory) but “in” the point. We are interested in our research in the diagrams which play an *optical* role – microscopes and “microscopes within microscopes”, telescopes, windows, a *mirror* role (to externalize rough mental models), and an *unveiling* role (to help create new and interesting mathematical concepts, theories, and structures). In this paper we describe some examples of optical diagrams as a particular kind of *epistemic mediator* able to perform the explanatory abductive task of providing a better understanding of the calculus, through a non-standard model of analysis. We also maintain they can be used in many other different epistemological and cognitive situations.

The Explanatory and Abductive Function of Mathematical Diagrams

More than a hundred years ago, the great American philosopher Charles Sanders Peirce used the term “abduction” to refer to inference that involves the generation and evaluation of explanatory hypotheses. Peirce says that mathematical and geometrical reasoning “consists in constructing a diagram according to a general precept¹, in observing certain relations between parts of that diagram not explicitly required by the precept, showing that these relations will hold for all such diagrams, and in formulating this conclusion in general terms. All valid necessary reasoning is in fact thus diagrammatic” (Peirce, 1958, *CP*, 1.54). We contend that a considerable part of scientific reasoning is a kind of abductive reasoning.

What is abduction? Many reasoning conclusions that do not proceed in a deductive manner are *abductive*. For instance, if we see a broken horizontal

glass on the floor we might explain this fact by postulating the effect of wind shortly before: this is certainly not a deductive consequence of the glass being broken (a cat may well have been responsible for it). Hence, abduction (Magnani, 2001) is the process of *inferring* certain facts and/or laws and hypotheses that render some sentences plausible, that *explain* or *discover* some (eventually new) phenomenon or observation; it is the process of reasoning in which explanatory hypotheses are formed and evaluated.

Following Nersessian (1995a, 1995b), we use the term “model-based reasoning” to indicate the construction and manipulation of various kinds of representations, not mainly sentential and/or formal, but mental and/or related to external models. Obvious examples of model-based reasoning are constructing and manipulating visual representations, thought experiment, analogical reasoning, occurring when models are built at the intersection of some operational interpretation domain – with its interpretation capabilities – and a new ill-known domain, for example, in mathematical reasoning.

Peirce gives an interesting example of a simple model-based abduction related to sense activity: “A man can distinguish different textures of cloth by feeling: but not immediately, for he requires to move fingers over the cloth, which shows that he is obliged to compare sensations of one instant with those of another” (Peirce, 1958, *CP*, 5.221). This idea surely suggests that abductive movements also have interesting extra-theoretical characteristics and that there is a role in abductive reasoning for various kinds of manipulations of external objects. When manipulative aspects of external models prevail, like in the case of manipulating diagrams in the blackboard, we face what we call *manipulative* abduction (or action-based abduction).

Manipulative abduction happens when we are thinking *through* doing and not only, in a pragmatic sense, about doing. For instance, when we are creating geometry constructing and manipulating a triangle. In the case of natural sciences the idea of manipulative abduction goes beyond the well-known role of experiments as capable of forming new scientific laws by means of the results (nature’s answers to the investigator’s question) they present, or of merely

¹That is a kind of definition that prescribes “what you are to *do* in order to gain perceptual acquaintance with the object of the world” (Peirce, 1958, *CP*, 2.330).

playing a predictive role (in confirmation and in falsification).

It is indeed interesting to note that in mathematics model-based and manipulative abductions are present. For example, geometrical constructions present situations that are curious and “at the limit”. These are constitutively dynamic, artificial, and offer various contingent ways of epistemic acting, like looking from different perspectives, comparing subsequent appearances, discarding, choosing, re-ordering, and evaluating. Moreover, they present some of the features indicated below, typical of the so-called abductive *epistemic mediators* (Magnani, 2001): simplification of the task and the capacity to get visual information otherwise unavailable.

Epistemic mediators exhibit very interesting features (for example, we can find the first three in geometrical constructions): 1. action elaborates a *simplification* of the reasoning task and a redistribution of effort across time (Hutchins, 1995), when we need to manipulate concrete things in order to understand structures which are otherwise too abstract, or when we are in the presence of *redundant* and unmanageable information; 2. action can be useful in the presence of *incomplete* or *inconsistent* information – not only from the “perceptual” point of view – or of a diminished capacity to act upon the world: it is used to get more data to restore coherence and to improve deficient knowledge; 3. action enables us to build *external artifactual models* of task mechanisms instead of the corresponding internal ones, that are adequate to adapt the environment to agent’s needs. 4. action as a *control of sense data* illustrates how we can change the position of our body (and/or of the external objects) and how to exploit various kinds of prostheses (Galileo’s telescope, technological instruments and interfaces) to get various new kinds of stimulation: action provides some tactile and visual information (e.g. in surgery), otherwise unavailable.

Diagrams serve an important role in abduction because they can be manipulated. In mathematics diagrams play various roles in a typical abductive way. Two of them are central:

- they provide an intuitive and mathematical *explanation* able to help the understanding of concepts difficult to grasp or that appear obscure and/or epistemologically unjustified. We will present in the following sections some new diagrams (microscopes within microscopes), which provide new mental representations of the concept of tangent line at the infinitesimally small regions.
- they help *create* new previously unknown concepts, as illustrated in the case of the discovery of the non-Euclidean geometry in (Magnani, 2002).

Mirror, Unveiling, and Optical Diagrams as External Representations

Certainly a big portion of the complex environment of a thinking agent is internal, and consists of the proper software composed of the knowledge base and of the inferential expertise of that individual. Nevertheless, any cognitive system consists of a “distributed cognition” among people and “external” technical artifacts (Hutchins, 1995, Norman, 1993).

In the case of the construction and examination of diagrams in mathematics (for example in geometry), specific experiments serve as states and the implied operators are the manipulations and observations that transform one state into another. The mathematical outcome is dependent upon practices and specific sensory-motor activities performed on a non-symbolic object, which acts as a dedicated external representational medium supporting the various operators at work. There is a kind of an epistemic negotiation between the sensory framework of the mathematician and the external reality of the diagram. This process involves an external representation consisting of written symbols and figures that are manipulated “by hand”. The cognitive system is not merely the mind-brain of the person performing the mathematical task, but the system consisting of the whole body (cognition is *embodied*) of the person plus the external physical representation. For example, in geometrical discovery the whole activity of cognition is located in the system consisting of a human together with diagrams.

An external representation can modify the kind of computation that a human agent uses to reason about a problem: the Roman numeration system eliminates, by means of the external signs, some of the hardest parts of the addition, whereas the Arabic system does the same in the case of the difficult computations in multiplication (Zhang, 1997). The capacity for inner reasoning and thought results from the internalization of the originally external forms of representation. In the case of the external representations we can have various objectified knowledge and structure (like physical symbols – e.g. written symbols, and objects – e.g. three-dimensional models, shapes and dimensions), but also external rules, relations, and constraints incorporated in physical situations (spatial relations of written digits, physical constraints in geometrical diagrams and abacuses) (Zhang, 1997). The external representations are contrasted to the internal representations that consist of the knowledge and the structure in memory, as propositions, productions, schemas, neural networks, models, prototypes, images.

The external representations are not merely memory aids: they can give people access to knowledge and skills that are unavailable to internal representations, help researchers to easily identify aspects and to make further inferences, they constrain the range

of possible cognitive outcomes in a way that some actions are allowed and other forbidden. The mind is limited because of the restricted range of information processing, the limited power of working memory and attention, the limited speed of some learning and reasoning operations; on the other hand the environment is intricate, because of the huge amount of data, real time requirement, uncertainty factors. Consequently, we have to consider the whole system, consisting of both internal and external representations, and their role in optimizing the whole cognitive performance of the distribution of the various sub-tasks (Trafton et al., 2002). It is well-known that in the history of geometry many researchers used internal mental imagery and mental representations of diagrams, but also self-generated diagrams (external) to help their thinking.

In the construction of mathematical concepts many external representations are exploited, both in terms of diagrams and of symbols. We are interested in our research in diagrams which play an *optical* role – microscopes (that look at the infinitesimally small details), telescopes (that look at infinity), windows (that look at a particular situation), a *mirror* role (to externalize rough mental models), and an *unveiling* role (to help create new and interesting mathematical concepts, theories, and structures).²

Optical diagrams play a fundamental explanatory (and didactic) role in removing obstacles and obscurities and in enhancing mathematical knowledge of critical situations. They facilitate new internal representations and new symbolic-propositional achievements. In the example studied in the following section in the area of the calculus, the extraordinary role of the optical diagrams in the interplay standard/non-standard analysis is emphasized. Some of them could also play an unveiling role, providing new light on mathematical structures: it can be hypothesized that these diagrams can lead to further interesting creative results. The optical and unveiling diagrammatic representation of mathematical structures activates *direct perceptual operations* (for example identifying how a real function appears in its points and/or to infinity; how to really reach its limits).

We stated that in mathematics diagrams play various roles in a typical abductive way (cf. the previous section). Now we can add that:

- they are *epistemic mediators* able to perform various abductive tasks in so far as
- they are *external representations* which, in the cases we will present in the following sections, are devoted to providing explanatory abductive results.

²The epistemic and cognitive role of mirror and unveiling diagrams in the discovery of non-Euclidean geometry is illustrated in (Magnani, 2002).

Perceiving the Infinite and the Infinitesimal World in Calculus

The concept of tangent line of a real function is normally based on the standard ε, δ concept of limit, which is intrinsically difficult to represent and not immediately assimilable, for example, by students (see (Sullivan, 1976)). We can avoid this trouble by introducing a pictorial device that allows a better understanding by the visualization of small details in the graph of a curve $y = f(x)$. This method was invented by Stroyan (1972) and improved by Tall (1982, 2001): our intention is to continue and improve Tall's work by applying it to many other different situations. We will work on the hyperreal number system \mathbb{R}^* and will assume the non-standard analysis given by Abraham Robinson (1966).³

In the present and in the following section we will explain the method and the classification proposed by Tall. In the last two sections we will introduce new types of diagrams called microscopes “within” microscopes. Then, we will provide an example to show how difficulties can be avoided through this type of diagram.

By visualizing the difference between the numbers a and $a + \varepsilon$ (where $a \in \mathbb{R}$ and ε is a positive infinitesimal), we can introduce the map $\mu : \mathbb{R}^* \rightarrow \mathbb{R}^*$ given by

$$\mu(x) = \frac{x - a}{\varepsilon}.$$

Thus $\mu(a) = 0$ and $\mu(a + \varepsilon) = 1$, that is, μ maps a and $a + \varepsilon$, two infinitely close points, onto clearly distinct points 0 and 1. We may also identify, through μ , a point a with its corresponding $\mu(a)$.

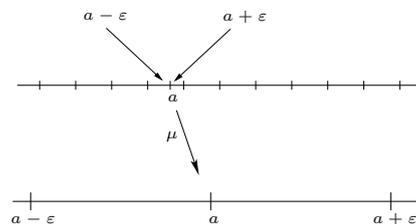


Figure 1: The hyperreal line and the map μ .

In general, for all $\alpha, \delta \in \mathbb{R}^*$, the function $\mu : \mathbb{R}^* \rightarrow \mathbb{R}^*$ given by

$$\mu(x) = \frac{x - \alpha}{\delta} \quad (\delta \neq 0)$$

is called δ -lens pointed at α . But what can we see through a lens? What kind of details can it reveal? We define *field of view* of μ the set of $x \in \mathbb{R}^*$ such that $\mu(x)$ is finite. Given two infinitesimals ε, δ , we say that ε is of *higher order* than δ , *same order* as δ ,

³For an easy introduction to non-standard calculus see (Keisler, 1976a, 1976b).

or *lower order* than δ if ε/δ is, respectively, infinitesimal, finite but not infinitesimal or infinite. It follows from this definition that, if ε is of higher order than δ , ε is an infinitesimal “smaller” than δ .

Given a δ -lens μ , proceeding by taking the standard part of μ , we obtain a function from the field of view in \mathbb{R} , called the *optical δ -lens pointed in α* . The optical lenses are actually what we need to visualize infinitesimal quantities. In fact, our eyes are able to distinguish clearly only images on the real plane \mathbb{R}^2 . As such, the optical δ -lens translate on the \mathbb{R}^2 plane, in favor of our eyes, everything that differs from α in the same order as δ . Higher order details are “too small” to see and lower order details are “too far” to capture within the field of view. Two points in the field of view that differ by a quantity of higher order than δ appear the same through the optical δ -lens.

This method also works in two coordinates (and, in general, in n coordinates) by the application of a lens to every coordinate. The map

$$\mu : \mathbb{R}^{*2} \rightarrow \mathbb{R}^{*2}, \quad \mu(x, y) = \left(\frac{x - \alpha}{\delta}, \frac{y - \beta}{\rho} \right)$$

is called *(δ, ρ)-lens pointed in (α, β)* . If $\delta \neq \rho$, we say that the lens is *astigmatic*. If $\delta = \rho$, we can talk about δ -lens in two dimensions. By considering the standard parts of every coordinate, we obtain an optical δ -lens in two dimensions, defined from the field of view of μ in \mathbb{R}^2 .

However, δ may not be infinitesimal. Depending on its nature, there are different kinds of lenses: if δ is infinitesimal, then the lens is called a *microscope*; if δ is finite but not infinitesimal, then the lens is a *window*; if δ is infinite, the lens is a *macroscope*. A window pointed at a point with at least one infinite coordinate is called a *telescope*.

Microscopes reveal infinitesimal details and telescopes allow us to visualize a structure at infinity. For example, through an optical microscope, a differentiable function looks like a straight line and through an optical telescope two asymptotic curves look identical.

Microscopes and Differentiable Functions

Now we can easily generalize Tall’s example about the role of microscopes (Tall, 1982, 2001). An infinitesimal increment Δx of a differentiable function f from its point x can be written as follows

$$f(x + \Delta x) = f'(x)\Delta x + f(x) + \varepsilon\Delta x$$

where ε is infinitesimal. Thus, we can fix $(a, f(a))$ on the graph and point on it an optical Δx -lens to magnify infinitesimal details that are too small to see to the naked eye. We have

$$\mu(x, y) = \left(\frac{x - a}{\Delta x}, \frac{y - f(a)}{\Delta x} \right).$$

An infinitely close point $(a + \lambda, f(a + \lambda))$, when viewed through μ , becomes

$$\mu(a + \lambda, f(a + \lambda)) = \left(\frac{\lambda}{\Delta x}, \frac{f'(a)\lambda + \lambda\varepsilon}{\Delta x} \right).$$

Suppose that λ is of the same order as Δx , i.e. $\lambda/\Delta x$ is finite. This means that $\lambda\varepsilon/\Delta x$ is infinitesimal. By taking the standard parts, we have

$$\begin{aligned} \left(\text{st} \left(\frac{\lambda}{\Delta x} \right), \text{st} \left(\frac{f'(a)\lambda}{\Delta x} + \frac{\lambda\varepsilon}{\Delta x} \right) \right) &= \\ &= \left(\text{st} \left(\frac{\lambda}{\Delta x} \right), f'(a) \text{st} \left(\frac{\lambda}{\Delta x} \right) \right). \end{aligned}$$

If a is fixed, putting $\text{st}(\lambda/\Delta x) = t$, we see that the points on the graph in the field of view are mapped in the straight line $(t, f'(a)t)$, where t varies (see Figure 2). Note that the slope of the line is, in effect, the derivative of f in the point a and the function is really indistinguishable from its tangent in an infinitesimal neighborhood of a .

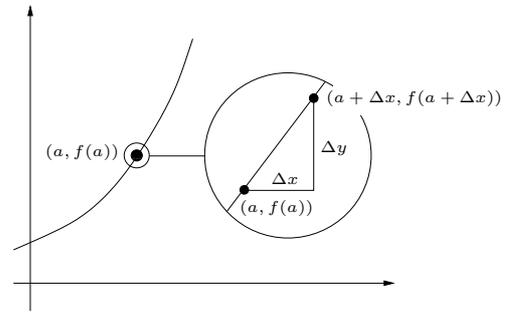


Figure 2: A graph of a differentiable function through an optical Δx -lens.

In the following sections we will describe some interesting new mathematical situations in which such lenses can be used to construct a suitable mental representation.

Microscopes “within” Microscopes

This type of diagram was originally suggested and used by Keisler (1976a, 1976b), but not formalized by constructing optical lenses.

Let f be a real function with continuous second derivative ($f \in C^2$). If we magnify an infinitesimal neighborhood by a more powerful tool than an optical Δx -lens, we can see other interesting properties of the curve. This is what we call a microscope “within” a microscope pointed in $(a + \Delta x, f(a + \Delta x))$ in the non-optical Δx -lens (because the optical lenses lose every infinitesimal details). By an optical Δx -lens pointed in $(a, f(a))$, both the curve $y = f(x)$ and the tangent $y = f'(a)(x - a) + f(a)$ are

mapped in the line $(t, f'(a)t)$, where $t = \text{st}(\lambda/\Delta x)$ and λ is an infinitesimal of the same order as Δx . Now we can put $\lambda = \Delta x$ and point a Δx^2 -lens in $(a + \Delta x, f(a + \Delta x))$. In order to visualize more details, we need to have more information about the function: our idea is to use the non-standard Taylor's second order formula for f (see (Stroyan and Luxemburg, 1976)), i.e.

$$f(a + \Delta x) = f(a) + f'(a)\Delta x + \frac{1}{2}f''(a)\Delta x^2 + \varepsilon_1\Delta x^2$$

where ε_1 is infinitesimal.

Thus the Δx^2 -lens maps as follows

$$(x, y) \mapsto \left(\frac{x - (a + \Delta x)}{\Delta x^2}, \frac{y - f(a + \Delta x)}{\Delta x^2} \right)$$

and the point $(a + \Delta x, f(a + \Delta x))$ is mapped onto $(0, 0)$. Let λ be an infinitesimal of the same order as Δx^2 . The Taylor's second order formula gives

$$f(a + \Delta x + \lambda) = f(a) + f'(a)(\Delta x + \lambda) + \frac{1}{2}f''(a)(\Delta x + \lambda)^2 + \varepsilon_2(\Delta x + \lambda)^2.$$

Therefore, we have

$$\begin{aligned} (a + \Delta x + \lambda, f(a + \Delta x + \lambda)) &\mapsto \\ \left(\frac{\lambda}{\Delta x^2}, \frac{f(a + \Delta x + \lambda) - f(a + \Delta x)}{\Delta x^2} \right) &= \\ = \left(\frac{\lambda}{\Delta x^2}, \frac{f'(a)\lambda + \frac{1}{2}f''(a)\lambda^2 + f''(a)\Delta x\lambda}{\Delta x^2} \right. \\ &\left. + \frac{\varepsilon_2\Delta x^2 + \varepsilon_2\lambda^2 + 2\varepsilon_2\Delta x\lambda - \varepsilon_1\Delta x^2}{\Delta x^2} \right) \end{aligned}$$

and by taking the standard parts

$$\left(\text{st} \left(\frac{\lambda}{\Delta x^2} \right), f'(a) \text{st} \left(\frac{\lambda}{\Delta x^2} \right) \right)$$

as the other terms are all infinitesimals.

The point $(a + \Delta x + \lambda, f'(a)(\Delta x + \lambda) + f(a))$ on the graph of the tangent line is mapped in the point

$$\begin{aligned} \left(\frac{\lambda}{\Delta x^2}, \frac{f'(a)(\Delta x + \lambda) - f'(a)\Delta x}{\Delta x^2} \right. \\ \left. + \frac{-\frac{1}{2}f''(a)\Delta x^2 - \varepsilon_1\Delta x^2}{\Delta x^2} \right) &= \\ = \left(\frac{\lambda}{\Delta x^2}, \frac{\lambda f'(a) - \frac{1}{2}f''(a)\Delta x^2 - \varepsilon_1\Delta x^2}{\Delta x^2} \right) &= \\ = \left(\frac{\lambda}{\Delta x^2}, f'(a) \frac{\lambda}{\Delta x^2} - \frac{1}{2}f''(a) - \varepsilon_1 \right) \end{aligned}$$

and then the optical lens gives

$$\left(\text{st} \left(\frac{\lambda}{\Delta x^2} \right), f'(a) \text{st} \left(\frac{\lambda}{\Delta x^2} \right) - \frac{1}{2}f''(a) \right).$$

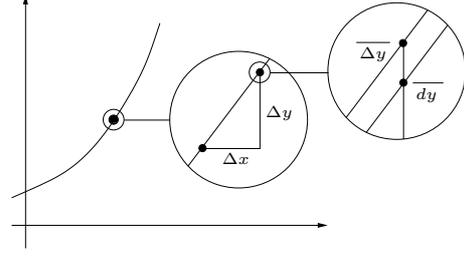


Figure 3: A microscope “within” a microscope.

This suggests nice new (and mathematically justified, of course) mental representations of the concept of tangent line: through the optical Δx^2 -lens, the tangent line can be seen as the line $(t, f'(a)t - \frac{1}{2}f''(a))$ which means that the graph of the function and the graph of the tangent are distinct, straight, and parallel lines in a Δx^2 -neighborhood of $(a + \Delta x, f(a + \Delta x))$. The fact that one line is either below or above the other, depends on the sign of $f''(a)$, in accordance with the standard real theory: if $f''(x)$ is positive (or negative) in a neighborhood, then f is convex (or concave) here and the tangent line is below (or above) the graph of the function.

A Cognitive Application of Microscopes within Microscopes

In this section we will show how a diagram easily allows the construction of a mathematical concept.

We saw that through a microscope within a microscope a curve and its tangent are respectively

$$y(t) = f'(a)t \quad \text{and} \quad y(t) = f'(a)t - \frac{1}{2}f''(a).$$

Then, what happens when $f \in C^2$ is such that $f''(a) = 0$, for example when a is an inflection point for f ? In this case the second microscope would still show the tangent line indistinguishable from the curve (see Figure 4). What does this mean? We can simply deduce that in an inflection point a curve that is differentiable two times has a particular behavior: here it is very slightly curved and much more similar to a straight line (its tangent). An expert mathematician would say that it has a small *curvature*. In fact, the curvature of a function at a point t of its domain is the quantity defined by

$$\frac{|f''(t)|}{(1 + (f'(t))^2)^{3/2}}$$

and it is a value of how much the curve locally differs from the tangent line. For example, a straight line has null curvature and a circle has constant curvature.

In an inflection point, a function $f \in C^2$ has curvature equal to 0. In other words, in this point the

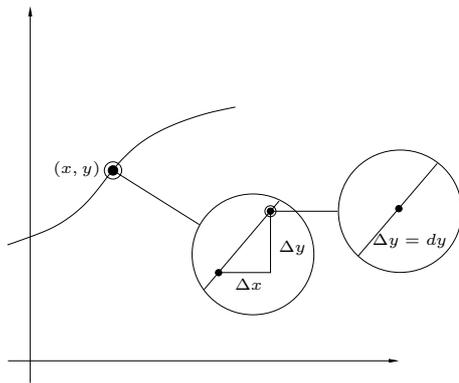


Figure 4: An inflection point through a microscope within microscope.

graph is much more than simply indistinguishable from its tangent, it has a more marked “straight local trend”. In order to discover this property in standard calculus, the concept of curvature is necessary. On the contrary, the simpler idea of microscope within microscope allows to discover the same property immediately, easily and without the concept of curvature.

Conclusion

The optical diagrams we have described provide explanations which allow a better understanding of calculus. They also improve and complete the non-standard method given by Abraham Robinson: they are necessary tools for it, both from the psychological (didactic) and the epistemological point of view, because they propose a good – and mathematically justified – mental representation of the behavior of a real function in many “critical” situation (in small neighborhoods, at infinity, by looking at infinitesimally small details . . .).

Moreover: i) the role of optical diagrams in a calculus teaching environment seems relevant. We are preparing experimental research on the calculus students at the University of Pavia (mathematics and engineering curricula) devoted to detecting the details of the didactic effects and the learning improvements; ii) we are convinced they can be exploited in other everyday non-mathematical applications (finding routes, road signs, buildings maps, for example), in connection to various zooming effects of spatial reasoning; iii) we think the activity of magnification of optical diagrams can be studied in other areas of model-based reasoning, such as the ones involving creative, analogical, and spatial inferences, both in science and everyday situations so that this can extend the psychological theory.

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